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Anomaly-free Multiple Singularity Enhancement in F-theory

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Abstract: We study global Calabi-Yau realizations of multiple singularity enhancement relevant for family-unification model building in F-theory. We examine the conditions under which the generation of extra chiral matter at multiple singularities on 7-branes in six-dimensional F-theory can be consistent with anomaly cancellation. It is shown that the generation of extra matter is consistent only if it is accompanied by simultaneous degenerations of loci of the leading polynomial of the discriminant so that the total number of chiral matter does not change. We also show that the number of singlets expected to arise matches the decrease of the complex structure moduli for the restricted geometry.

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1 Introduction

In [1], it was pointed out that multiple singularities on 7-branes in F-theory [2] may serve as the basis for a realization of the coset sigma model spectrum relevant for "family unification" [3–12]. The key observation was that, in six dimensions, the representation of chiral matter localized at an enhanced split-type singularity [13, 14] is labeled by some homogeneous Kähler manifold, the reason for which was explained [15, 16] by investigating the string junctions [17–27] near the singularity. Applying the same argument to the singularities where multiple matter branes simultaneously intersect the gauge 7-branes, it was argued that the chiral matter hypermultiplet spectrum at such a multiple singularity consists of those that form a homogeneous Kähler manifold with more than one $U(1)$ factors in the denominator of the coset.

In this paper, we examine whether this chiral matter spectrum at such a multiple singularity is consistent with the absence of anomalies of the theory[1]. In six dimensions the condition for anomaly cancellation imposes a severe restriction on the chiral matter spectrum [29–31].[2] At first sight, it seems that the matter spectrum corresponding to a coset with multiple $U(1)$ factors conflicts with anomaly cancellation since it needs to be accompanied by generation of extra chiral hypermultiplets. We will, however, show

[1] Singularity enhancement with rank more than one was considered in the extensive study [28] on singularities and matter representations.

[2] The anomaly analysis has also been useful for the study of six-dimensional conformal field theories (See e.g. [32–36]).

that such a coset spectrum is indeed possible without ruining the balance of anomalies. Rather, in some cases when the complex structure moduli take certain values, the absence of anomalies *requires* that there must occur such generation of extra matter at the multiple singularity. Although we work in the F-theory compactifications on elliptic Calabi-Yau threefolds over a Hirzebruch surface $\mathbf{F}_n$ [13, 14], the best understood example of an F-theory compactification, the mechanism we find is local and will apply to other compactifications on elliptic Calabi-Yau manifolds.

In the next section we recall what representations of hypermultiplets are expected to arise at a multiple singularity. In section 3 we review the anomaly cancellation mechanisms for $\mathcal{N} = 1$, $D = 6$ supersymmetric theories. We will also see there that in the case of six-dimensional F-theory on an elliptic CY3 over $\mathbf{F}_n$, which is known to be dual to $E_8 \times E_8$ heterotic string on $K3$, no net increase of chiral matter is allowed either by the ordinary heterotic Green-Schwarz mechanism or by the generalized Green-Schwarz mechanism first applied by Sadov. In section 4, we give examples of 7-brane configurations which include some multiple singularities *but* the number of hypermultiplets in each representation *does not change* compared with the generic 7-brane configurations in the nearby moduli space, and hence the theory remains anomaly-free. As we will see, such a transition is possible if and only if it is accompanied by simultaneous degenerations of loci of the leading polynomial of the discriminant so that the necessary extra degrees of freedom at the singularity may be supplemented by the appropriate number of "extra-zero" loci joining there simultaneously. We will also show that the decrease of the dimensions of the moduli space for the special class of configurations matches the number of new singlets appearing at the multiple singularity, which is consistent with the anomaly cancellation.

2 Multiple singularity enhancement in F-theory in six dimensions

2.1 F-theory on an elliptic CY3 over $\mathbf{F}_n$

Let us recall the basic setting of the F-theory compactification on an elliptically fibered Calabi-Yau over a Hirzebruch surface $\mathbf{F}_n$ [13, 14]. The three-fold is defined by the Weierstrass equation:

$$y^2 = x^3 + f(z, z')x + g(z, z'), \tag{2.1}$$

$$f(z, z') = \sum_{i=0}^{8} z^i f_{8+(4-i)n}(z'), \tag{2.2}$$

$$g(z, z') = \sum_{i=0}^{12} z^i g_{12+(6-i)n}(z'). \tag{2.3}$$

A Hirzebruch surface $\mathbf{F}_n$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. z and z' is the coordinates of the fiber and the base, respectively. The coefficients $f_{8+(4-i)n}(z')$ $(i = 0, \ldots, 6)$ and $g_{12+(6-i)n}(z')$ $(i = 0, \ldots, 12)$ are polynomials of z' of degrees specified by the subscripts. Both x and y are complex, so the equation (2.1) determines some torus at each (z, z'). More precisely, x,y,f and g are sections of $\mathcal{L}^2, \mathcal{L}^3, \mathcal{L}^4$ and $\mathcal{L}^6$, where $\mathcal{L}$ is the anti-canonical line bundle of

the base $\mathbf{F}_n$. The total space is then an elliptic Calabi-Yau threefold, which is also a $K3$ fiberation over the $\mathbb{P}^1$ parameterized by z'.

In order to illustrate what kind of singularity we are interested in, let us first consider a concrete example. Suppose that the coefficient polynomials of the lower order terms in the expansions of $f(z,z')$ (2.2) and $g(z,z')$ (2.3) take the particular forms:

$$\begin{aligned}
f_{4n+8} &= -3h_{n+2}^4, \\
f_{3n+8} &= 12h_{n+2}^2 H_{n+4}, \\
f_{2n+8} &= 12\left(h_{n+2}q_{n+6} - H_{n+4}^2\right), \\
g_{6n+12} &= 2h_{n+2}^6, \\
g_{5n+12} &= -12h_{n+2}^4 H_{n+4}, \\
g_{4n+12} &= 12h_{n+2}^2(2H_{n+4}^2 - h_{n+2}q_{n+6}), \\
g_{3n+12} &= -f_{n+8}h_{n+2}^2 + 24h_{n+2}H_{n+4}q_{n+6} - 16H_{n+4}^3, \\
g_{2n+12} &= -f_8 h_{n+2}^2 + 2f_{n+8}H_{n+4} + 12q_{n+6}^2
\end{aligned} \tag{2.4}$$

for some polynomials h_{n+2},H_{n+4} and q_{n+6}; they are so arranged that the discriminant starts with the z^5 term to produce a $I_5 = SU(5)$ Kodaira singularity [37] along the line $z = 0$. For later convenience we present an explicit form of the lower order expansions of this curve in appendix A.

The independent polynomials preserving this particular singularity structure are

$$h_{n+2},\ H_{n+4},\ q_{n+6},\ f_{n+8} \text{ and } g_{n+12}. \tag{2.5}$$

The total degrees of freedom is thus

$$(n+3)+(n+5)+(n+7)+(n+9)+(n+13)-1 = 5n+36, \tag{2.6}$$

which matches the number of $SU(5)$ singlets computed by using the index theorem on the heterotic side[3].

Since the leading order term of the discriminant Δ is

$$\begin{aligned}
\Delta &= 108z^5 h_{n+2}^4 P_{3n+16} + \cdots, \\
P_{3n+16} &\equiv -2f_8 h_{n+2}^2 H_{n+4} - 2f_{n+8}h_{n+2}q_{n+6} + f_{8-n}h_{n+2}^4 + g_{n+12}h_{n+2}^2 - 24H_{n+4}q_{n+6}^2
\end{aligned} \tag{2.7}$$

the singularity gets enhanced to a higher one at the $n+2$ zero loci of h_{n+2} and the $3n+16$ loci of P_{3n+16}[4].

At the zero loci of a *generic* P_{3n+16} (so that $h_{n+2} \neq 0$, in particular), the order of Δ becomes ≥ 6 while ord$f$ and ord$g$ remain zero. If ord$\Delta = 6$, the singularity is enhanced to

[3] Note that the "middle" coefficients f_8, g_{12} and the higher ones $f_{8-n}, \ldots; g_{12-n}, \ldots$ are not counted here as the complex structure moduli which are to be compared with the singlets arising form "this" E_8 factor. This is because the middle ones f_8, g_{12} correspond to the geometric moduli of the elliptic $K3$ while the higher ones are taken into account in the similar analysis for the singularity at $z = \infty$ corresponding to the other (partially broken) E_8 gauge factor.

[4] Although it contains f_8 and f_{8-n}, they only affect the positions of the loci and do not affect the total number of the loci.

$I_6 = SU(6)$ and a chiral matter in **5** appears at each zero locus of P_{3n+16}. On the other hand, at the $n+2$ loci of h_{n+2}, the first few terms of f and g simultaneously vanish so that f starts with z^2 and g does with z^3, as long as H_{n+4} does not vanish there. Also, the order of the discriminant becomes 7. This is the $I_5^* = SO(10)$ singularity, and the chiral matter is **10** at each zero of h_{n+2} for generic P_{3n+16}. In all, the matter spectrum for the generic $SU(5)$ curve is

$$(n+2)\mathbf{10}, \quad (3n+16)\mathbf{5}, \quad (5n+36)\mathbf{1}. \tag{2.8}$$

Originally [14] what kind of charged matter should appear at these enhanced "extra zeroes" was determined by referring to the massless spectrum of the dual heterotic model [13], that is, the $K3$ compactification of the $E_8 \times E_8$ heterotic string with instanton numbers $(12-n, 12+n)$. The relationship between the extra zeroes of the discriminant and the massless charged matter was first explained by Katz and Vafa [15] by mapping the problem to that of deformations of the singularities of $K3$. Later it was proposed by one of the present authors [16] how the chiral matter spectrum is understood by investigating string junctions near the enhanced singularity.

Spectral cover, matter localization and the Mordell-Weil group

One of the remarkable features of heterotic/F-theory duality is that a brane-like object naturally comes into play in heterotic theory through the construction of a vector bundle over the elliptic Calabi-Yau manifold [38]. Basically, the statement of heterotic/F-theory duality is made in a certain limit in the moduli space on both sides: F-theory is compactified on a $K3$-fibered Calabi-Yau where the $K3$ goes to a stable degeneration limit into two, themselves elliptically fibered, dP_9's intersecting along a two torus E, and heterotic string theory is on an elliptically fibered Calabi-Yau whose fiber torus has a large volume and the same complex structure as E. The moduli space of the vector bundle over each torus is known as Looijgenha's weighted projective space; for an $SU(5)$ gauge group this is an ordinary projective space. The *spectral cover* is a polynomial equation of x and y, the variables in the Weierstrass equation describing a heterotic torus fiber. The defining polynomial has five (for $SU(5)$) zero loci (which add up to zero) on the torus, each of which specifies a Wilson line of a Cartan generator and coordinatizes Looijgenha's projective space. In the $SU(5)$ case, the polynomial is explicitly [38]:

$$w = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y \tag{2.9}$$

for some coefficients $a_0, \ldots, a_5$.

On the other hand, we consider a pencil [39]

$$(y^2 + x^3 + \alpha_1 xyz + \alpha_2 x^2 z^2 + \alpha_3 y v^3 + \alpha_4 x v^4 + \alpha_6 v^6) + p(v,x,y)u = 0, \tag{2.10}$$
$$p(v,x,y) \;=\; a_0 v^5 + a_2 x v^3 + a_3 y v^2 + a_4 x^2 v + a_5 x^2 y \tag{2.11}$$

in $\mathbf{WP}^3_{(1,1,2,3)}$ with the equivalence relation $(u,v,x,y) \sim (\lambda u, \lambda v, \lambda^2 x, \lambda^3 y)$, $\lambda \in \mathbf{C}$. Obviously, $p(v,x,y)$ (2.11) is the homogenization of w (2.9). After blowing up $u = v = 0$ the

pencil (2.10) becomes dP_9, which we regard as one of two dP_9's appearing in the stable degeneration limit on the F-theory side. Indeed, we can show that if we set

$$\begin{aligned}
a_5 &= 2\sqrt{3}u^{-1}h_{n+2},\\
a_4 &= u^{-1}(\sqrt{3}\alpha_1 h_{n+2}+6H_{n+4}),\\
a_3 &= -\sqrt{3}u^{-1}\left(\tfrac{1}{6}(\alpha_1^2-4\alpha_2)h_{n+2}+4q_{n+6}\right),\\
a_2 &= u^{-1}\left(\sqrt{3}(-\tfrac{1}{12}\alpha_1^3+\tfrac{1}{3}\alpha_1\alpha_2+\alpha_3)h_{n+2}+(-\alpha_1^2+4\alpha_2)H_{n+4}-2\sqrt{3}\alpha_1 q_{n+6}+f_{n+8}\right),\\
a_0 &= u^{-1}\left(\tfrac{\sqrt{3}}{12}\alpha_3\left(4\alpha_2-\alpha_1^2\right)h_{n+2}+(2\alpha_4-\alpha_1\alpha_3)H_{n+4}-2\sqrt{3}\alpha_3 q_{n+6}+\tfrac{1}{12}(4\alpha_2-\alpha_1^2)f_{n+8}-g_{n+12}\right),
\end{aligned} \tag{2.12}$$

the pencil (2.10) precisely reproduces the lower terms (up to the "middle" ones) of the $SU(5)$ Weierstrass equation (2.1)(2.2)(2.3) with (2.4). Therefore, the polynomials (2.5) of [14] correspond to

$$h_{n+2}\sim a_5,\quad H_{n+4}\sim a_4,\quad q_{n+6}\sim a_3,\quad f_{n+8}\sim a_2,\quad g_{n+12}\sim a_0. \tag{2.13}$$

Furthermore, it was also shown by using the Leray spectral sequence [40, 41] that the matter is localized where some of a_j's vanish and some of the zero loci of w (or $p(v,x,y)$) go to infinity. We note that this may be intuitively understood as a consequence of the structure theorem of the Mordell-Weil group [42, 43]. Indeed, the equation $p(v,x,y)=0$ defines sections of dP_9, and since the structure theorem [44, 45] states the singularities and the sections are orthogonal complement of each other in E_8, the less sections we have, the more singularities we get instead.

The Mordell-Weil lattice was studied in detail in terms of string junctions in [23] using the isomorphism between the string junction algebra and the Picard lattice of a rational elliptic surface. For a recent F-theory phenomenological aspect of the Mordell-Weil group see [46, 47].

2.2 Multiple singularity enhancement from $SU(5)$ to E_6

We will now consider what happens if h_{n+2} and H_{n+4} simultaneously vanish. In this case, the z^2 term of f and the z^3 term of g vanish at these points, and the order of the discriminant rises up to 8. This means that the singularity gets enhanced from $I_5=SU(5)$ to $IV^*=E_6$ there.

Note that if $h_{n+2}=0$, that H_{n+4} vanishes means that P_{3n+16} also does. Thus this higher singularity can be viewed as a consequence of a collision of an $I_1^*=SO(10)$ singularity, occurring at a zero of h_{n+2}, and an $I_6=SU(6)$ singularity, which corresponds to a zero of P_{3n+16}.

In the standard 7-brane representation of the Kodaira singularity, the $SU(5)$ singularity is made of a collection of five **A**-branes, while the $SO(10)$ singularity is represented by $\mathbf{A}^5\mathbf{BC}$. Thus the zero loci of the polynomial h_{n+2} are the places where a **B**- and a **C**-branes intersect the five **A**-branes lying on top of each other (FIG.1, left). On the other hand, if H_{n+4} happens to vanish at the same point, then the singularity becomes E_6 which

is represented by $\mathbf{A}^5\mathbf{BCC}$. Therefore, this multiple singularity occurs when an extra **C**-brane simultaneously meets the five **A**-branes together in addition to the **B**- and **C**-branes (FIG.1, right).

However, suppose that we slightly move from this special point in the complex structure moduli space to another where h_{n+2} and H_{n+4} do *not* simultaneously vanish but the roots of $h_{n+2} = 0$ and $P_{3n+16} = 0$ are still close (Note that if h_{n+2} is not zero, $H_{n+4} = 0$ does not mean $P_{3n+16} = 0$.) This will correspond to the split of the multiple E_6 singularity into an $SO(10)$ singularity and an $SU(6)$ singularity. While it is OK for the pair of **B**- and **C**-branes to form the D_5 singularity, how can the remaining **C**-brane yield the A_5 singularity with the five **A**-branes?

This apparent contradiction can be explained as follows: We should first note that *any* isolated discriminant locus has monodromy

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.14}$$

and hence, locally, is identified as a location of an **A**-brane. However, when this discriminant locus merges with a D_5 singularity to form an E_6 singularity, a pair of zero loci of g and f also joins with the discriminant locus since the orders of g and f are respectively enhanced by one.

To understand why the **C**-brane can produce the A_5 singularity, we must know the monodromies around the zero loci of f and g. The zero locus of f is mapped, by the inverse

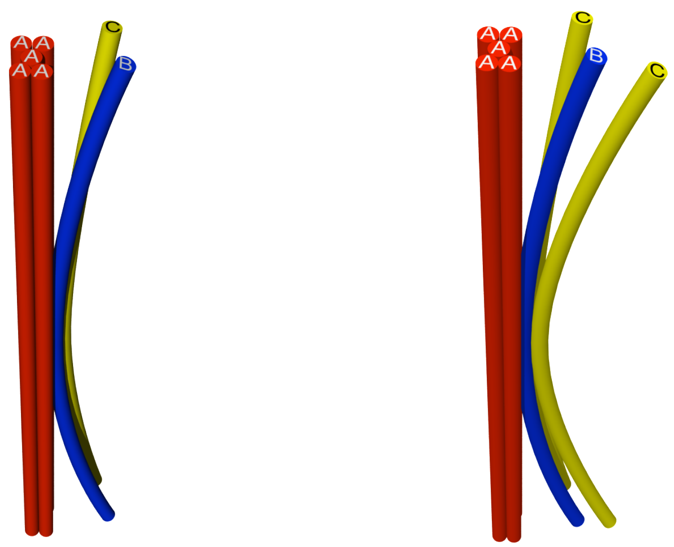

Figure 1. Left: D_5 singularity. Right: E_6 singularity.

J-function via the relation

$$J(\tau) = \frac{4f^3}{4f^3 + 27g^2}, \tag{2.15}$$

to (taking the starting point in Im$J > 0$ in the standard fundamental region of the modular group) $\tau = e^{\frac{2\pi i}{3}}$, near which $J(\tau)$ behaves like

$$J(\tau) = (\tau - e^{\frac{2\pi i}{3}})^3(1 + O(\tau - e^{\frac{2\pi i}{3}})). \tag{2.16}$$

That is, if J changes its value along a small closed path encircling 0 *three times*, τ goes around $e^{\frac{2\pi i}{3}}$ precisely once, back to the original fundamental region; this can be verified by tracing the value of the J function [16]: Since $J \simeq \mathrm{const}.f^3$ near $f = 0$, if one goes around the zero locus of f once counter-clockwise on the z plane, the value of J goes around zero three times counter-clockwise. Therefore, the monodromy around the locus of f is

$$\begin{aligned}(ST^{-1})^3 &= -1 \\ &\simeq 1 \quad \text{in } PSL(2, \mathbb{Z}),\end{aligned} \tag{2.17}$$

and hence is identity as a modular transformation. Here

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.18}$$

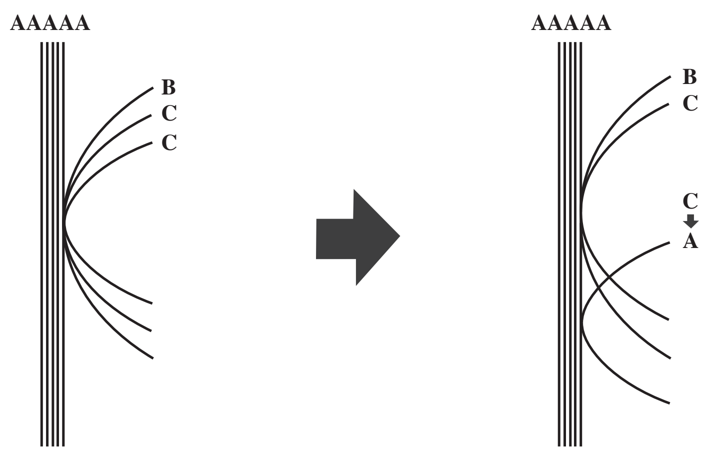

Figure 2. An E_6 singularity is split into a D_5 and an A_5 singularities.

Likewise, the zero locus of g is mapped to $\tau = i$, and the expansion of $J(\tau)$ is then

$$J(\tau) = 1 + (\tau - i)^2 (1 + O(\tau - i)). \tag{2.19}$$

So if the value of J goes around 1 *twice*, τ then does around i once, again back to the original fundamental region. Since $J - 1 \simeq \text{const.} g^2$ near $g = 0$, circling around the zero locus of g once on the z plane means that the corresponding τ circles around i once. Thus the monodromy around the locus of g reads

$$\begin{aligned} (S^{-1})^2 &= -1 \\ &\simeq 1 \quad \text{in } PSL(2, \mathbb{Z}), \end{aligned} \tag{2.20}$$

and again is identity.

Now let us consider the effect of these loci of f and g to the monodromy of the other coalescing 7-branes. As we discussed above, any discriminant locus is locally an **A** brane. However, when this and the $\mathbf{A}^5\mathbf{BC}$ branes come close to merge, it turns out that there is also a locus of g situated in between them. So if the reference point of the monodromy is set near the $\mathbf{A}^5\mathbf{BC}$ branes, then one undergoes the S^{-1} transformation when one passes

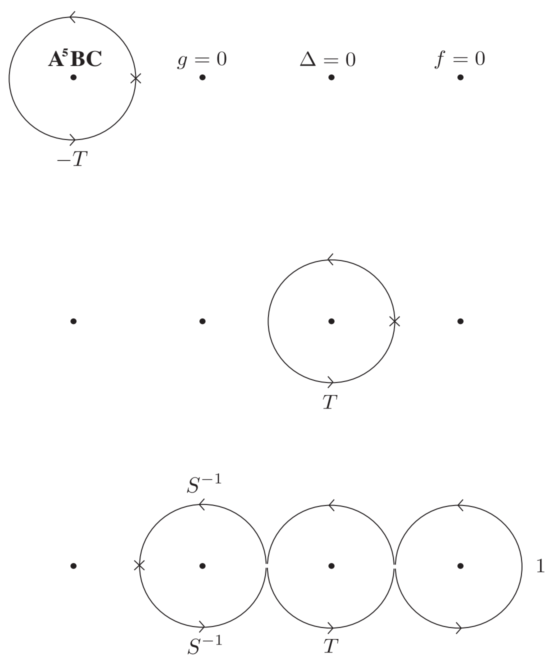

Figure 3.

by the locus of g. Therefore, the monodromy of the discriminant locus is

$$S^{-1}TS^{-1} \tag{2.21}$$

(FIG. 3), which is equal to

$$-T^{-1}CT \simeq T^{-1}CT \quad \text{in } PSL(2,\mathbb{Z}), \tag{2.22}$$

where C is the monodromy matrix of $\mathbf{C}$:

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}. \tag{2.23}$$

Since the monodromy matrix of $\mathbf{A}^5\mathbf{BC}$ is $= -T$ which is invariant under T conjugation, we see that this discriminant locus can merge with the D_5 singularity as a **C** brane. If, on the other hand, the **B** and **C** branes split off from the coalesced five **A** branes and one "**C** brane", the locus of g also splits off, and therefore what has been a **C** brane in the E_6 singularity turns into an **A** brane, yielding the A_5 singularity together with the five **A** branes.

As we saw in the previous section, if the **A** brane alone meets the gauge 7-branes while the **B** and **C** branes are apart, then this singularity corresponds to one of the roots of $P_{3n+16} = 0$ and the charged matter **5** appears. These BPS states are thought of as coming from string junctions connecting the **A** brane and the gauge branes localized near the intersection point. Likewise, if only the **B** and **C** branes intersect while the **A** brane is apart, the singularity is of $SO(10)$ and a **10** will arise due to the string junctions connecting the the **B** and **C** branes and the gauge branes; this happens at the loci of h_{n+2}. Therefore, if $P_{3n+16} = 0$ and h_{n+2} are simultaneously zero, then there will arise both **10** and **5** at that point. The former comes from the string junctions $a_i + a_j - b - c$ $(1 \le i < j \le 5)$ while the latter can be identified as the ones of the form $\sum_{k=1}^{5} a_k - a_i - 2b - c - c'$ (TABLE I)[5].

But we also notice that, at this E_6 point, there are not only these two kinds of string junctions but still more BPS junctions: the ones of the form $a_i + a_j - b - c'$ $(1 \le i < j \le 5)$ and $c - c'$. They are special BPS junctions that appear only at this higher singularity and are not present at generic points in the moduli space. Since they are BPS, these extra string junctions are also expected to give rise to chiral matter at the multiple singularity. This was the proposal of ref.[1].

However, there is a puzzle here: Suppose that the theory is at some generic point in the moduli space of an elliptic Calabi-Yau three-fold over a Hirzebruch surface where it is dual to some $E_8 \times E_8$ heterotic string compactification on $K3$. Of course, the theory is anomaly free. Then suppose that the values of the moduli parameters are tuned to some special ones so that the 7-branes develop a multiple singularity. If it gives rise to more chiral matter hypermultiplets than those present at a generic point in the moduli space, doesn't that conflict with anomaly cancellation?

In the next section, we will see that the requirement of anomaly cancellation severely limits the conditions under which this phenomenon consistently occurs.

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[5] For each matter locus there are in fact twice as many BPS junctions corresponding to overall $\pm$ multiplications.

3 Anomaly cancellation in six dimensions

The relevant anomaly eight-forms are given by[6]

$$(\hat{I}_{3/2}^{(D=6)})_8 = \frac{49}{36}Y_4 - \frac{43}{72}Y_2^2, \tag{3.1}$$
$$(\hat{I}_{1/2}^{0})_8 = \frac{1}{180}Y_4 + \frac{1}{72}Y_2^2, \tag{3.2}$$
$$(\hat{I}_{1/2}^{L_i})_8 = \mathrm{dim}L_i\left(\frac{1}{180}Y_4 + \frac{1}{72}Y_2^2\right) + \frac{1}{16}\left(\frac{2}{3}\mathrm{tr}_{L_i}F^4 - \frac{1}{6}\mathrm{tr}_{L_i}F^2\mathrm{tr}R^2\right), \tag{3.3}$$
$$(\hat{I}_A)_8 = \frac{7}{45}Y_4 - \frac{1}{9}Y_2^2, \tag{3.4}$$

where

$$Y_{2m} \equiv \frac{1}{2}\left(-\frac{1}{4}\right)^m \mathrm{tr}R^{2m}, \tag{3.5}$$

and L_i denotes the representation of the unbroken gauge group G.

In general, the total anomaly polynomial is given by

$$\begin{aligned}\hat{I}_8^{\mathrm{total}} = &-\left((\hat{I}_{3/2}^{(D=6)})_8 + (\hat{I}_A)_8\right) + n_T\left((\hat{I}_A)_8 + (\hat{I}_{1/2}^0)_8\right) - \sum_\alpha (\hat{I}_{1/2}^{\mathrm{Ad}G_\alpha})_8 \\ &+\sum_i n_H^i(\hat{I}_{1/2}^{L_i})_8 \quad + n_H^0(\hat{I}_{1/2}^0)_8,\end{aligned} \tag{3.6}$$

where n_T is the number of tensor multiplets, n_H^i is the number of massless hypermultiplets in the representation L_i of the unbroken gauge group and n_H^0 is the number of other neutral hypermultiplets not counted in n_H^i as singlets. We assume that the unbroken gauge group is a direct product $\prod_\alpha G_\alpha$. We write

$$\begin{aligned} n_V &\equiv \sum_\alpha \mathrm{dim}G_\alpha, \\ n_H &\equiv \sum_i n_H^i \mathrm{dim}L_i + n_H^0, \end{aligned} \tag{3.7}$$

then if they satisfy the well-known relation: .7

$$n_H - n_V = 273 - 29n_T, \tag{3.8}$$

the $\mathrm{tr}R^4$ terms cancel out and we have [48]

$$\begin{aligned}\hat{I}_8^{\mathrm{total}} = &\frac{9-n_T}{2}Y_2^2 - \frac{1}{12}Y_2\sum_\alpha(\mathrm{Tr}_\alpha F_\alpha^2 - \sum_i n_H^{\alpha i}\mathrm{tr}_{L_i^\alpha}F_\alpha^2) \\ &-\frac{1}{24}\sum_\alpha(\mathrm{Tr}_\alpha F_\alpha^4 - \sum_i n_H^{\alpha i}\mathrm{tr}_{L_i^\alpha}F_\alpha^4) \\ &+\frac{1}{4}\sum_{\alpha<\beta}\sum_{i,j} n_H^{\alpha i;\beta j}\mathrm{tr}_{L_i^\alpha}F_\alpha^2\mathrm{tr}_{L_j^\beta}F_\beta^2,\end{aligned} \tag{3.9}$$

[6] They are $-16\pi^4$ times the ones given in [29].

where, as usual, Tr_α denotes the trace taken in the adjoint representation of G_α. $n_H^{\alpha i}$ is the number of hypermultiplets in the representation L_i^α of G_α, and $n_H^{\alpha i;\beta j}$ is one in $L_i^\alpha \otimes L_j^\beta$ of $G_\alpha \times G_\beta$.

It is known [13, 14] that F-theory compactified on an elliptic Calabi-Yau three-fold over the the Hirzebruch surface $\mathbf{F}_n$ is dual to the $K3$ compactification of the $E_8 \times E_8$ heterotic string with instanton numbers $(12+n, 12-n)$, so let us first recall the perturbative spectrum of the $K3$ compactifications of the $E_8 \times E_8$ heterotic string.

Let $H^{(m)}$ $(m = 1, 2)$ be the gauge group of instanton in $E_8^{(m)}$ with instanton number $12 + (-1)^{m-1}n$, and $G^{(m)}$ be the maximal commutant in $E_8^{(m)}$. Let the decomposition of the adjoint of E_8 in the representations of $G^{(m)} \times H^{(m)}$ be

$$\mathbf{248}^{(m)} = \oplus_i (L_i^{(m)} \otimes C_i^{(m)}). \tag{3.10}$$

for each $m = 1, 2$. Let $F_0^{(m)}$ $(m = 1, 2)$ be the field strength of the instanton in $H^{(m)}$, and define $r_i^{(m)}$ as the ratio of the traces

$$\mathrm{tr}_{C_i^{(m)}} F_0^{(m)2} = r_i^{(m)} \mathrm{Tr}_{E_8} F_0^{(m)2}. \tag{3.11}$$

Then the number of hypermultiplets are given by the index theorem:

$$\begin{aligned} -\, n_H^0 &= -21, \\ -n_H^{(m)\alpha i} &= \mathrm{dim} C_i^{(m)} - \frac{1}{8\pi^2} \int_{K3} \frac{1}{2} \mathrm{tr}_{C_i^{(m)}} F_0^{(m)2}. \end{aligned} \tag{3.12}$$

Using these expressions in (3.9), one can show that [29]

$$\hat{I}_8^{\mathrm{total}} = 4 \left(Y_2 + \frac{1}{8}(x^{(1)} + x^{(2)}) \right) \left(Y_2 + \frac{n}{16}(x^{(1)} - x^{(2)}) \right), \tag{3.13}$$

where $x^{(m)} = \frac{1}{30} \mathrm{Tr}_{E_8} F^{(m)2}$ $(m = 1, 2)$. Thus the anomaly of the K3 compactification of the $E_8 \times E_8$ heterotic string factorizes and hence can be canceled by the Green-Schwarz mechanism.

In F-theory, an alternative anomaly cancellation mechanism is known: The generalized Green-Schwarz mechanism assumes that [48] the same anomaly polynomial (3.9) can be written in a bilinear form

$$\hat{I}_8^{\mathrm{total}} = \frac{1}{2} \Omega_{\hat{i}\hat{j}} X^{\hat{i}} X^{\hat{j}}, \tag{3.14}$$

$$X^{\hat{i}} \equiv \frac{1}{2} a^{\hat{i}} \mathrm{tr} R^2 + \sum_\alpha 2 b_\alpha^{\hat{i}} \mathrm{tr} F_\alpha^2 \tag{3.15}$$

for some constants $\Omega_{\hat{i}\hat{j}}$, $a^{\hat{i}}$ and $b_\alpha^{\hat{i}}$, where the repeated indices $\hat{i}$, $\hat{j}$ are understood to be summed over 1 through the total number of B fields. The anomaly is then written as

$$\int \Omega_{\hat{i}\hat{j}} \omega_2^{1\hat{i}} X^{\hat{j}} \tag{3.16}$$

with

$$X^{\hat{i}} = d\omega_3^{\hat{i}}, \tag{3.17}$$
$$\delta_\Lambda \omega_3^{\hat{i}} = d\omega_2^{1\hat{i}}(\Lambda), \tag{3.18}$$

which can be canceled by the contribution from the counterterm

$$\int \Omega_{\hat{i}\hat{j}} B^{\hat{i}} X^{\hat{j}}, \tag{3.19}$$

assuming that the anomalous transformations of the $B^{\hat{i}}$ fields

$$\delta_\Lambda B^{\hat{i}} = -\omega_2^{1\hat{i}}(\Lambda). \tag{3.20}$$

The conditions for the anomaly polynomial to be written in the form (3.14) are summarized by the following set of equations:

$$9 - n_T = \sum_{\hat{i},\hat{j}} \Omega_{\hat{i}\hat{j}} a^{\hat{i}} a^{\hat{j}}, \tag{3.21}$$
$$\mathrm{index}(\mathrm{Ad}G_\alpha) - \sum_{\hat{i}} n_H^{\alpha i} \mathrm{index}(L_i^\alpha) = 6 \sum_{\hat{i},\hat{j}} \Omega_{\hat{i}\hat{j}} a^{\hat{i}} b_\alpha^{\hat{j}} \tag{3.22}$$
$$x_{\mathrm{Ad}G_\alpha} - \sum_{\hat{i}} n_H^{\alpha i} x_{L_i^\alpha} = 0, \tag{3.23}$$
$$y_{\mathrm{Ad}G_\alpha} - \sum_{\hat{i}} n_H^{\alpha i} y_{L_i^\alpha} = -3 \sum_{\hat{i},\hat{j}} \Omega_{\hat{i}\hat{j}} b_\alpha^{\hat{i}} b_\alpha^{\hat{j}}, \tag{3.24}$$
$$\sum_{\hat{i},\hat{j}} n_H^{\alpha i;\beta j} \mathrm{index}(L_i^\alpha) \mathrm{index}(L_j^\beta) = \sum_{\hat{i},\hat{j}} \Omega_{\hat{i}\hat{j}} b_\alpha^{\hat{i}} b_\beta^{\hat{j}}, \tag{3.25}$$

where, following [48], we have defined

$$\mathrm{tr}_{L_i^\alpha} F_\alpha^2 = \mathrm{index} L_i^\alpha \ \mathrm{tr}_\alpha F_\alpha^2, \tag{3.26}$$
$$\mathrm{tr}_{L_i^\alpha} F_\alpha^4 = x_{L_i^\alpha} \mathrm{tr}_\alpha F_\alpha^4 + y_{L_i^\alpha} (\mathrm{tr}_\alpha F_\alpha^2)^2 \tag{3.27}$$

for some trace tr_α taken in a preferred representation of G_α. In the following we take the fundamental representation for this representation for $SU(N)$ or $SO(2N)$, **27** for E_6, **56** for E_7 and **248** for E_8.

The anomaly (3.13) is also canceled by this mechanism. Indeed, (3.13) is further written in a compact form:

$$\hat{I}_8^{\mathrm{total}} = \frac{1}{32} \left(\frac{1}{2} K \mathrm{tr} R^2 + D_u x^{(1)} + D_v x^{(2)} \right)^2, \tag{3.28}$$

where K is the canonical divisor of the Hirzebruch surface $\mathbf{F}_n$, and D_u, D_v are the divisors of the sections $z = 0, \infty$, respectively. The square on the right hand side is understood as

an intersection product. By choosing the divisor of the fiber D_s and D_v above as a basis, K, D_u and D_v can be expressed in terms of component vectors:

$$\begin{aligned} K &= -(2+n)D_s - 2D_v \;\equiv\; K^{\hat{i}} D_{\hat{i}}, \\ D_u &= nD_s + D_v \;\equiv\; D_u^{\hat{i}} D_{\hat{i}}, \\ D_v &\equiv D_v^{\hat{i}} D_{\hat{i}}. \end{aligned} \tag{3.29}$$

The intersection form is given by

$$\Omega_{\hat{i}\hat{j}} = \begin{pmatrix} D_s{\cdot}D_s & D_s{\cdot}D_v \\ D_v{\cdot}D_s & D_v{\cdot}D_v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}. \tag{3.30}$$

Then $\hat{I}_8^{\rm total}$ can be written in the form (3.14) with

$$X^{\hat{i}} = \frac{1}{2} a^{\hat{i}} trR^2 + \sum_{m=1,2} \sum_{G_\alpha \in E_8^{(m)}} 2 b_\alpha^{(m)\hat{i}} {\rm tr}_\alpha F_\alpha, \tag{3.31}$$

$$a^{\hat{i}} = K^{\hat{i}}, \tag{3.32}$$

$$b_\alpha^{(1)\hat{i}} = \frac{1}{60 r_\alpha} D_u^{\hat{i}}, \tag{3.33}$$

$$b_\alpha^{(2)\hat{i}} = \frac{1}{60 r_\alpha} D_v^{\hat{i}}, \tag{3.34}$$

where

$$r_\alpha \equiv \frac{{\rm tr}_\alpha F_\alpha^2}{{\rm Tr}_{E_8} F_\alpha^2}. \tag{3.35}$$

Thus F-theory on an elliptic Calabi-Yau over $\mathbf{F}_n$, which shares the same matter spectrum as that of $E_8 \times E_8$ heterotic string on $K3$ with instanton numbers $(12+n, 12-n)$, can be anomaly-free also by this anomaly cancellation mechanism.

3.1 Multiple singularities and anomalies

As we already mentioned, the F-theory compactification on an elliptic Calabi-Yau over $\mathbf{F}_n$, is dual to the $E_8 \times E_8$ heterotic string compactification on $K3$ with instanton numbers $(12+n, 12-n)$. More precisely, suppose that $G \times H$ is a direct product maximal subgroup of, say, the first factor of E_8, such that (1) G is simple and simply-laced, and (2) H is semi-simple. We assume that H has $12+n$ instantons so the unbroken gauge group from this E_8 is G. For these cases the massless spectra of heterotic string and the dual geometries of F-theory are summarized in appendix B. As it is shown there, for each such pair (G, H), one can find a specialized Weierstrass form of the elliptic fiberation such that

(1) The total number of the deformation parameters of the curve that preserve the particular singularity structure is precisely equal to the number of neutral hypermultiplets (arising from this E_8) computed by the index theorem in heterotic string theory.

(2) The leading order term in z of the discriminant of the Weierstrass form factorizes, and the degree in z' of each factor again coincides with the number of charged hypermultiplets obtained by the index theorem.

Therefore, at least for this class of F-theory compactifications, there is no anomaly since they share the same massless matter contents as those of heterotic strings on $K3$. This is the situation where the both theories are at "generic" points in the moduli spaces. On the other hand, suppose that the F-theory curve is deformed in such a way that more than one factor of the leading order term of the discriminant comes to share a common zero locus, at which the singularity is more enhanced than the ones occurring at the ordinary matter loci at generic points in the moduli space. As we discussed at the end of the previous section, such a multiple singularity supports more BPS string junctions than when the discriminant loci are split apart. Since for all the cases listed in TABLE 3 a half of the localized string junctions at the enhanced point precisely correspond to the matter representations predicted from the heterotic string analysis, one may also expect that the additional string junctions appearing at the multiple point will also play a role to the generation of massless matter in F-theory.

However, the anomaly cancellation condition forbids such net increase of chiral matter. For the Green-Schwarz mechanism to work, the total anomaly polynomial must factorize into the form (3.13) or (3.14). This requires the absence of the Y_4 term in $\hat{I}_8^{\rm total}$, and that imposes the constraint:

$$n_H - n_V = 273 - 29 n_T, \tag{3.36}$$

as we saw previously. Thus as long as the number of the tensor multiplets is one, as is so for the smooth heterotic compactifications, the only possible change in the number of the hypermultiplets is one associated with the simultaneous change in the number of the vector multiplets, that is, the Higgs mechanism. In the present case, however, there is no gauge symmetry enhancement to expect at the multiple singularity, so this does not happen.

Also, even if one allows to change the number of tensor multiplets, the anomalies from the net increase of the hypermultiplets cannot be canceled. This is because the coefficient of n_T in (3.8) is *minus* 29, and hence the increase of n_T means the *decrease* of the hypermultiplets.

The total number of chiral matter is also constrained by geometry. In the generalized Green-Schwarz mechanism, the change in the number of the tensor hypermultiplets means the change in the self-intersection number of the canonical class K of the base manifold of the elliptic fiberation; see (3.21). In the present case, the base is a Hirzebruch surface. The canonical class can change if the surface is blown up at some points. Suppose that the Hirzebruch surface is blown up at a point, the canonical class is changed to [49]

$$\begin{aligned} K &= -(2+n)D_s - 2D_v \\ &\to -(2+n)D_s - 2D_v + e_1, \end{aligned} \tag{3.37}$$

where e_1 is the exceptional divisor that has arisen due to the blow up. Since its intersection pairing is

$$e_1 \cdot e_1 = -1, \quad D_s \cdot e_1 = D_v \cdot e_1 = 0, \tag{3.38}$$

the self-intersection $K \cdot K$ *decreases* from eight to seven, which also implies that there arises more tensor multiplets and less hypermultiplets are allowed to exist.[7]

Therefore, in any case, any net change of the total number of chiral matter is inconsistent with anomaly cancellation. Is there any transition of geometry without any change of the total number of hypermultiplets before and after the transition? In fact, an example of such a transition to special points in the moduli space has already been found in [16], where the branes have some multiple singularities and at the same time the theory remains anomaly free. We will discuss this in the next section.

4 Anomaly-free multiple singularities

4.1 Enhancement from $SU(5)$ to $SO(12)$

The curve found in [16] is one which has an $SU(5) = I_5$ singularity at $z = 0$, and also parameterized by (2.4), except that q_{n+6} is further specialized to the form

$$q_{n+6} = h_{n+2} q_4 \tag{4.1}$$

for some fourth-order polynomial q_4 in z'.[8] This means that all the roots of the equation h_{n+2} are also ones of q_{n+6}. In this particular case we have

$$\begin{aligned}
f(z,z') &= -3h_{n+2}^4 + 12 z h_{n+2}^2 H_{n+4} + z^2 \left(12 q_4 h_{n+2}^2 - 12 H_{n+4}^2\right) + z^3 f_{n+8} + \cdots , \\
g(z,z') &= 2h_{n+2}^6 - 12 z h_{n+2}^4 H_{n+4} + z^2 h_{n+2}^2 \left(24 H_{n+4}^2 - 12 q_4 h_{n+2}^2\right) \\
&\quad + z^3 \left(-f_{n+8} h_{n+2}^2 + 24 q_4 h_{n+2}^2 H_{n+4} - 16 H_{n+4}^3\right) + z^4 \left(2 f_{n+8} H_{n+4} + 12 q_4^2 h_{n+2}^2\right) \\
&\quad + z^5 g_{n+12} + \cdots , \\
\Delta &= 108 z^5 h_{n+2}^6 \left(-2 q_4 f_{n+8} + g_{n+12} - 24 q_4^2 H_{n+4}\right) \\
&\quad -9 z^6 h_{n+2}^4 \left(-96 q_4 f_{n+8} H_{n+4} + f_{n+8}^2 + 72 g_{n+12} H_{n+4} + 96 q_4^3 h_{n+2}^2 - 1152 q_4^2 H_{n+4}^2\right) \\
&\quad +36 z^7 h_{n+2}^2 (30 q_4^2 f_{n+8} h_{n+2}^2 - 24 q_4 f_{n+8} H_{n+4}^2 + f_{n+8}^2 H_{n+4} - 18 q_4 g_{n+12} h_{n+2}^2 \\
&\qquad\qquad + 36 g_{n+12} H_{n+4}^2 + 432 q_4^3 h_{n+2}^2 H_{n+4} - 288 q_4^2 H_{n+4}^3) \\
&\quad -18 z^8 (3 f_{n+8} g_{n+12} h_{n+2}^2 - 72 q_4^2 f_{n+8} h_{n+2}^2 H_{n+4} - 8 q_4 f_{n+8}^2 h_{n+2}^2 + 2 f_{n+8}^2 H_{n+4}^2 \\
&\qquad - 72 q_4 g_{n+12} h_{n+2}^2 H_{n+4} + 48 g_{n+12} H_{n+4}^3 - 216 q_4^4 h_{n+2}^4) + \cdots .
\end{aligned} \tag{4.2}$$

We see that the coefficient of the leading order term of Δ has been changed to the form:

$$\begin{aligned}
\Delta &= 108 z^5 h_{n+2}^6 P_{n+12} + \cdots , \\
P_{n+12} &\equiv -2 q_4 f_{n+8} + g_{n+12} - 24 q_4^2 H_{n+4}
\end{aligned} \tag{4.3}$$

[7] Such a transition was first considered in [50]. Colliding singularities in F-theory on a blown-up Hirzebruch were studied in [51].

[8] Examples of multiple singularity enhancement from $SU(5)$ to $SO(12)$, E_6 or E_7 were more recently considered in [28].

from $h_{n+2}^4 P_{3n+16}$ (2.7) for the generic $SU(5)$ curve. If h_{n+2} vanishes, then f and g start from $O(z^2)$ and $O(z^3)$, respectively, and Δ vanishes all the way up to $O(z^7)$ with $O(z^8)$ being the first nonvanishing term. This is a $D_6 = I_2^*$ singularity, which means that the curve has $n+2$ points with multiple singularity enhancement $SU(5) \to SO(12)$.

Note that this is another case of a collision of the loci of h_{n+2} and P_{3n+16} discussed in section II. To see this we set $h_{n+2} = 0$ in P_{3n+16} (2.7) to find that

$$P_{3n+16} \sim -24 H_{n+4} q_{n+6}^2. \tag{4.4}$$

Thus if either of H_{n+4} or q_{n+6} vanishes, there occurs a collision [9]. The former case was discussed in section II, where the singularity was enhanced to E_6[10] .

In the present case, the BPS junctions are the ones corresponding to the homogeneous Kähler manifold $SO(12)/(SU(5) \times U(1)^2)$:

$$\mathbf{10}(SO(10)) \oplus \mathbf{10}(SU(5)) = \mathbf{5} \oplus \bar{\mathbf{5}} \oplus \mathbf{10} \tag{4.5}$$

plus one $\mathbf{1}$ from the extra Cartan subalgebra. Since $\mathbf{5}$ and $\bar{\mathbf{5}}$ are indistinguishable in six dimensions, we have

$$\mathbf{5} \oplus \mathbf{5} \oplus \mathbf{10} \oplus \mathbf{1} \tag{4.6}$$

residing at each zero of h_{n+2} ($SO(10)$ point). Thus the hypermultiplets at the brane intersections are

$$(n+2)\,(\mathbf{5} \oplus \mathbf{5} \oplus \mathbf{10} \oplus \mathbf{1}) \oplus (n+12)\mathbf{5} = (n+2)\mathbf{10} \oplus (3n+16)\mathbf{5} \oplus (n+2)\mathbf{1}, \tag{4.7}$$

where the $(n+12)$ $\mathbf{5}$'s on the left hand side come from the zeros of P_{n+12} (4.3). In addition, there are singlets from the complex structure moduli; their number is determined by the degrees of freedom of the polynomials

$$h_{n+2},\ H_{n+4},\ q_4,\ f_{n+8} \text{ and } g_{n+12}, \tag{4.8}$$

which yield

$$(n+3) + (n+5) + 5 + (n+9) + (n+13) - 1 = 4n + 34 \tag{4.9}$$

more $\mathbf{1}$'s, and hence $5n+36$ singlets in all. Thus the matter spectrum coincides with (2.8) and hence is unchanged from that for the generic unbroken $SU(5)$ curve we saw in section II, and therefore the theory remains anomaly-free!

How can this happen despite the extra $\mathbf{5}$ at each zero locus of h_{n+2}? We can see this by noticing that the degree of the other factor of the leading term of the discriminant is changed to $n+12$ from $3n+16$ for the generic case. That is, $2n+4$ of $3n+16$ loci

[9] This collision is not the kind of one that needs a blowup on the base, unlike the cases discussed in [51]. An extra tensor multiplet would make the theory anomalous in the present case as we saw at the end of the previous section.

[10] As we will see below, the fact that q_{n+6} is being squared is important since it means a simultaneous degeneration of two loci of P_{3n+16}.

of **5** have *pairwise degenerated* into $n+2$ pairs and simultaneously coalesced with the locus of **10** (FIG.4(a))! Thus the total number of charged matter is unchanged. It is also remarkable that the balance of the neutral hypermultiplets is maintained before and after the multiple singularity enhancement; the emergence of the $n+2$ extra singlets at the singularity is precisely compensated by the decreased amount of complex structure moduli for the restricted geometry [11].

The geometry considered in this section is the one with a *maximal* number of multiple enhanced points from $SU(5)$ to $SO(12)$; one may equally well consider the case where, for arbitrary integer r $(0 \leq r \leq n+2)$, $2r$ of $3n+16$ loci of **5** pairwise merge with r **10** loci while the rest of **5** loci remain as they are. It is easy to see also in this case the numbers of charged and neutral matter do not change before and after the coalesce of the singularities.

Conversely, if the extra matter did *not* arise at the multiple singularity enhancement with the simultaneous degeneration of matter loci as above, the balance of the matter multiplets (3.8) would be lost and the theory would become anomalous. Thus the absence of anomalies requires here the generation of extra matter at this multiple singularity.

4.2 Enhancement from $SU(5)$ to E_6

Having understood how an anomaly-free multiple singularity enhancement can be realized, we can now find curves with other types of multiple singularity enhancement. Let us reexamine in this section the singularity enhancement $SU(5) \rightarrow E_6$ considered in section II. As we saw there, this happens when h_{n+2} and H_{n+4} have a common zero locus.

We first examine the case when H_{n+4} takes the form

$$H_{n+4} = h_{n+2} H_2 \tag{4.10}$$

Although this particular form of H_{n+4} indeed creates $n+2$ E_6 points, the extra hypermultiplets such as those that were described at the end of section II do *not* arise. Indeed, with the form of H_{n+4} (4.10), the discriminant takes the form

$$\Delta = h_{n+2}^5 P_{2n+14} z^5 + \cdots , \tag{4.11}$$

but the E_6 multiple singularity with extra hypermultiplets is supposed to have two **10**'s at each locus so there are too many **10**'s to cancel anomalies.

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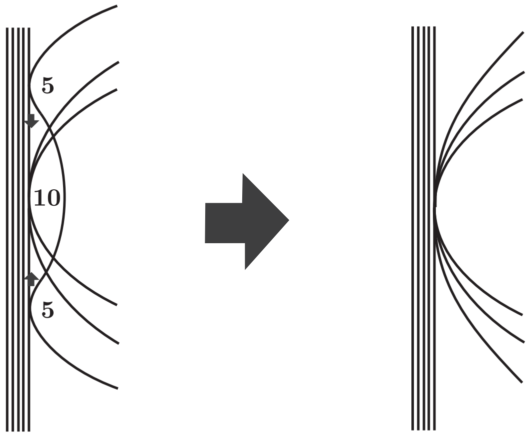
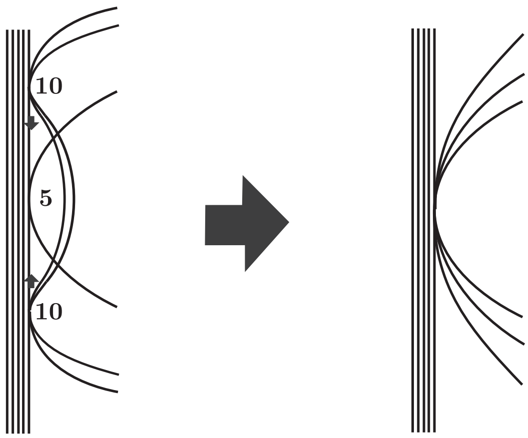
[11] A similar anomaly cancellation can be seen for $SU(3)$ curves with multiple singularities. The generic $SU(3)$ curve (see TABLE 2) has the discriminant $\Delta = h_{n+2}^3 P_{6n+18} z^3 + \cdots$. At each root of P_{6n+18}, the enhancement $I_3 \rightarrow I_4$ ($SU(3) \rightarrow SU(4)$) occurs and a **3** appears, giving in all $(6n+18)$**3**. At a root of h_{n+2}, the fiber type changes as $I_3 \rightarrow IV$ ($SU(3) \rightarrow SU(3)$), where a **B**-brane intersects the three **A**-branes. At this point, no extra BPS string junction can exist and hence no hypermultiplet appears. Specializing the generic curve in TABLE 2 to $H_{2n+6} = h_{n+2} q_{n+4}$, we obtain multiple singularities [16]. The discriminant changes to $\Delta = h_{n+2}^6 P_{3n+12} z^3 + \cdots$. It means that among the $6n+18$ roots of P_{6n+18}, $3n+6$ roots *triply degenerate* into $n+2$ sets and coalesce with zeros of h_{n+2}, yielding the multiple singularities. At the remaining $3n+12$ roots of P_{3n+12}, $(3n+12)$**3** appear. The decrease of $(3n+6)$**3** is precisely compensated by the **3**s at the multiple singularities of h_{n+2}. In fact, at each root of h_{n+2}, enhancement $I_3 \rightarrow I_0^*$ ($SU(3) \rightarrow SO(8)$) occurs and hypermultiplets in $\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1}$ appear. Note that this last **1** ($(n+2)$**1** in all) just compensates the decrease of $n+2$ neutral hypermultiplets due to the decrease of the complex structure moduli $H_{2n+6} \rightarrow q_{n+4}$ via the specialization.



(a)$SU(5) \to SO(12)$



(b)$SU(5) \to E_6$

Figure 4. Anomaly-free multiple singularities.

Therefore, to keep the number of **10** unchanged, we instead set

$$\begin{aligned} h_{n+2} &= h^2_{\frac{n+2}{2}}\,, \\ H_{n+4} &= h_{\frac{n+2}{2}} H_{\frac{n+6}{2}}\,. \end{aligned} \tag{4.12}$$

for some $h_{\frac{n+2}{2}}$, where $n+2$ is assumed to be divisible by two. With (4.12) the discriminant

reads

$$\Delta = h_{\frac{n+2}{2}}^{9} P_{\frac{5n+30}{2}} z^5 + \cdots . \tag{4.13}$$

We see that the $n+2$ roots of the equation $h_{n+2} = 0$ pairwise degenerate into $\frac{n+2}{2}$ double roots, each of which merges with a root of P_{3n+16} (FIG.4(b)).

The relevant homogeneous Kähler manifold in this case is $E_6/(SU(5) \times U(1)^2)$ whose $SU(5)$ representations are

$$\mathbf{16}(SO(10)) \oplus \mathbf{10}(SU(5)) = \mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1} \oplus \mathbf{10}, \tag{4.14}$$

and there is another $\mathbf{1}$ from the Cartan subalgebra. Thus the hypermultiplets coming from the brane intersections are

$$\tfrac{n+2}{2}\,(2 \cdot \mathbf{10} \oplus \mathbf{5} \oplus 2 \cdot \mathbf{1}) \oplus \tfrac{5n+30}{2}\mathbf{5} = (n+2)\mathbf{10} \oplus (3n+16)\mathbf{5} \oplus (n+2)\mathbf{1}. \tag{4.15}$$

On the other hand the decrease of the degrees of freedom of the polynomials is $\frac{n+2}{2}$ from $h_{n+2} \to h_{\frac{n+2}{2}}$ and $\frac{n+2}{2}$ from $H_{n+4} \to H_{\frac{n+6}{2}}$, in total $n+2$ again. This compensates the extra $n+2$ singlets in (4.15), and the theory after this transition also remains to be anomaly-free.

Although we have for simplicity considered the case with maximally possible multiple E_6 points with $n+2$ even, we may similarly consider the case with less E_6 points and/or $n+2$ odd. All that we require is that a pair of degenerating loci of h_{n+2} and a single locus of P_{3n+16} merge together per one E_6 multiple singularity. In this way the number of charged hypermultiplets is conserved, but again, the fact that the number of new singlets also matches the decrease of the complex structure moduli is rather nontrivial.

4.3 Enhancement from $SU(5)$ to E_7

Let us now consider a singularity enhancement in which the rank of the Lie algebra characterizing the singularity jumps up by more than two. The enhancement from $SU(5)$ to E_7 is of particular interest because it is relevant to the F-theory realization [1] of the Kugo-Yanagida $E_7/(SU(5) \times U(1)^3)$ family unification model [7].

The Kodaira classification tells us that the $E_7 = III^*$ singularity occurs when $\mathrm{ord} f = 3$, $\mathrm{ord} g \geq 5$ and $\mathrm{ord}\Delta = 9$. We can see from appendix A that this happens when h_{n+2}, H_{n+4} and q_{n+6} all simultaneously vanish. The homogeneous Kähler manifold for this singularity is $E_7/(SU(5) \times U(1)^3)$ with the following $SU(5)$ representations:

$$\begin{aligned}
\mathbf{27}(E_6) \oplus \mathbf{16}(SO(10)) \oplus \mathbf{10}(SU(5)) &= (\mathbf{16}(SO(10) \oplus \mathbf{10}(SO(10)) \oplus \mathbf{1}) \\
&\quad \oplus \mathbf{16}(SO(10)) \oplus \mathbf{10}(SU(5)) \\
&= 3 \cdot \mathbf{10} \oplus 4 \cdot \mathbf{5} \oplus 3 \cdot \mathbf{1},
\end{aligned} \tag{4.16}$$

where in the last line we have made no distinction between $\mathbf{5}$ and $\bar{\mathbf{5}}$. In addition, we have, this time, two $\mathbf{1}$'s from the Cartan subalgebra. In all, three $\mathbf{10}$'s, four $\mathbf{5}$'s and five singlets are supposed to arise at each multiple E_7 singularity. Thus, in order for the anomalies to cancel, we need to have three loci of h_{n+2} and four loci of P_{3n+16} to simultaneously degenerate and

join together, per one E_7 singularity. This is achieved, again for the maximal case, at the special points in the moduli space as follows:

$$\begin{aligned} h_{n+2} &= h^3_{\frac{n+2}{3}}\,, \\ H_{n+4} &= h^2_{\frac{n+2}{3}} H_{\frac{n+8}{3}}\,, \\ q_{n+6} &= h_{\frac{n+2}{3}} q_{\frac{2n+16}{3}}\,. \end{aligned} \tag{4.17}$$

for some $h_{\frac{n+2}{3}}$, $H_{\frac{n+8}{3}}$ and $q_{\frac{2n+16}{3}}$, where $n+2$ is assumed to be divisible by three in this case. The non-maximal case and/or the case in which $n+2$ is not 0 mod 3 are treated similarly. With (4.17) the discriminant becomes

$$\Delta = h^{16}_{\frac{n+2}{3}} P_{\frac{5n+40}{3}} z^5 + \cdots\,. \tag{4.18}$$

The total number of **5** is thus

$$4 \times \frac{n+2}{3} + \frac{5n+40}{3} = 3n + 16, \tag{4.19}$$

which is a correct value. Also the decrease of the degrees of freedom of the polynomials is

$$2 \times \frac{n+2}{3} + 2 \times \frac{n+2}{3} + 1 \times \frac{n+2}{3} = 5 \times \frac{n+2}{3}, \tag{4.20}$$

which match the five singlets residing at each of the $\frac{n+2}{3}$ E_7 points.

4.4 Enhancement from $SU(5)$ to E_8

The final example of anomaly-free singularity enhancement we consider in this paper is the one from $SU(5)$ to E_8. This type of multiple singularity may also be used for particle physics model building because the $D = 4$ supersymmetric nonlinear sigma model with $E_8/(SU(5) \times U(1)^4)$ as the target also yields net three chiral generations. Furthermore, it was pointed out [52] that this coset may also give rise to three sets of nonchiral singlet pairs needed in a scenario proposed by Sato and Yanagida [53] explaining the Yukawa hierarchies and large lepton-flavor mixings by the Frogatt-Nielsen mechanism [12].

The spectrum of $E_8/(SU(5) \times U(1)^4)$ is

$$5 \cdot \mathbf{10} \oplus 10 \cdot \mathbf{5} \oplus 10 \cdot \mathbf{1}. \tag{4.21}$$

With the additional three singlets from the Cartan subalgebra, in all

$$5 \cdot \mathbf{10} \oplus 10 \cdot \mathbf{5} \oplus 13 \cdot \mathbf{1} \tag{4.22}$$

reside at each E_8 point[13].

[12]In fact, the E_8 curve given in the original version of [52] did not take account of the simultaneous degenerations of loci and hence was anomalous as a six-dimensional theory. A revised version is in preparation.

[13]At first sight it seems that the $D = 4$, $\mathcal{N} = 1$ supersymmetric nonlinear sigma model with this target space may have five generations. However, one can show that [11] it is not possible to choose a so-called "Y-charge", a $U(1)$ charge that determines the complex structure of the coset space, in such a way that all the five "flavors" may have the same chirality. See [11] for more detail.

We can also find an anomaly-free curve with these E_8 multiple singularities. We again only present the case where $n+2$ is divisible by five and all the h_{n+2} loci turn into the E_8 singularities:

$$\begin{aligned} h_{n+2} &= h_{\frac{n+2}{5}}^5 , \\ H_{n+4} &= h_{\frac{n+2}{5}}^4 H_{\frac{n+12}{5}} , \\ q_{n+6} &= h_{\frac{n+2}{5}}^3 q_{\frac{2n+24}{5}} , \\ f_{n+8} &= h_{\frac{n+2}{5}}^2 f_{\frac{3n+36}{5}} \end{aligned} \tag{4.23}$$

for some $h_{\frac{n+2}{5}}$, $H_{\frac{n+12}{5}}$, $q_{\frac{2n+24}{5}}$ and $f_{\frac{3n+36}{5}}$. Then the discriminant reads

$$\Delta = h_{\frac{n+2}{5}}^{30} P_{n+12} z^5 + \cdots . \tag{4.24}$$

We can similarly verify that the numbers of both charged and neutral hypermultiplets are the same as those at generic points in the moduli space. Therefore, also in this case, the theory is anomaly-free.

It is interesting to notice that the powers of $h_{\frac{n+2}{5}}$ factors in (4.23) are precisely the exponents of $SU(5)$, that is, the powers of the canonical class projectivized in the weighted projective bundle [38], of which (2.13) are sections. Perhaps this coincidence may be interpreted in terms of spectral covers of the dual heterotic string theory.

5 Conclusions

We have shown that multiple singularity enhancement can really occur in F-theory without causing imbalance of anomalies. We have considered concrete examples in F-theory compactifications on an elliptically fibered Calabi-Yau over a Hirzebruch surface $\mathbf{F}_n$. Anomaly cancellation requires that there should be no net change in numbers of hypermultiplets after the coalesce of matter loci. We have presented such particular points in the moduli space in the case of unbroken $SU(5)$ gauge group, where the singularity is multiply enhanced to $SO(12)$, E_6, E_7 or E_8.

Although we have mainly considered the six-dimensional F-theory with $G = SU(5)$, it is natural to expect a similar anomaly-free transition to a configuration with multiple singularities to occur in four-dimensional compactifications and/or with other gauge groups. The original motivation to consider multiple singularity enhancement in F-theory was to construct “family unification” particle physics models in string theory. But if the number of chiral matter does not change after the coalesce of singularities, what is the use of the multiple singularities in string phenomenology model building?

To consider the multiple singularity enhancement in F-theory has at least three virtues:
(1) In general, a special point in the moduli space can be an end point of whatever flow in the moduli space after the supersymmetry is broken and potentials are generated; if it is not a special point, there is no reason for the flow to stop at that point.
(2) The multiple singularity may occur, in principle, in any elliptic Calabi-Yau manifold. Since the structure is universal, it may offer a potential ubiquitous mechanism for generating

three generations of flavors in the framework of F-theory.
(3) Last but not least, the homogeneous Kähler structure of the spectrum of the multiple singularity is naturally endowed with conserved $U(1)$ charges. This may also be useful for particle physics model building.

It would be extremely interesting to extend the analysis done in this paper to four dimensions.

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Appendix A $SU(5)$ curve

$$
\begin{aligned}
f(z,z') = & -3h_{n+2}^4 + 12zh_{n+2}^2 H_{n+4} + z^2\left(12h_{n+2}q_{n+6} - 12H_{n+4}^2\right) + z^3 f_{n+8} + f_8 z^4 + z^5 f_{8-n} \\
& + O\left(z^6\right), \qquad (5.1) \\
g(z,z') = & \, 2h_{n+2}^6 - 12z\, h_{n+2}^4 H_{n+4} \\
& + 12z^2 h_{n+2}^2 (2H_{n+4}^2 - h_{n+2}q_{n+6}) \\
& + z^3\left(-f_{n+8}h_{n+2}^2 + 24h_{n+2}H_{n+4}q_{n+6} - 16H_{n+4}^3\right) \\
& + z^4\left(-f_8 h_{n+2}^2 + 2f_{n+8}H_{n+4} + 12q_{n+6}^2\right) + z^5 g_{n+12} + g_{12}z^6 + O\left(z^7\right), \qquad (5.2) \\
\Delta = & \, 108z^5 h_{n+2}^4(-2f_8 h_{n+2}^2 H_{n+4} - 2f_{n+8}h_{n+2}q_{n+6} + f_{8-n}h_{n+2}^4 + g_{n+12}h_{n+2}^2 \\
& \quad - 24H_{n+4}q_{n+6}^2) \\
& - 9z^6 h_{n+2}^2(-96f_{n+8}h_{n+2}H_{n+4}q_{n+6} + 96f_{8-n}h_{n+2}^4 H_{n+4} - 144f_8 h_{n+2}^2 H_{n+4}^2 \\
& \quad + 24f_8 h_{n+2}^3 q_{n+6} + f_{n+8}^2 h_{n+2}^2 + 72g_{n+12}h_{n+2}^2 H_{n+4} - 12g_{12}h_{n+2}^4 + 96h_{n+2}q_{n+6}^3 \\
& \quad - 1152H_{n+4}^2 q_{n+6}^2) \\
& - 18z^7(-120f_8 h_{n+2}^3 H_{n+4}q_{n+6} + 48f_{n+8}h_{n+2}H_{n+4}^2 q_{n+6} - 144f_{8-n}h_{n+2}^4 H_{n+4}^2 \\
& \quad + 144f_8 h_{n+2}^2 H_{n+4}^3 - 2f_{n+8}^2 h_{n+2}^2 H_{n+4} + 48f_{8-n}h_{n+2}^5 q_{n+6} - 60f_{n+8}h_{n+2}^2 q_{n+6}^2 \\
& \quad + f_8 f_{n+8}h_{n+2}^4 + 36g_{12}h_{n+2}^4 H_{n+4} - 72g_{n+12}h_{n+2}^2 H_{n+4}^2 + 36g_{n+12}h_{n+2}^3 q_{n+6} \\
& \quad - 864h_{n+2}H_{n+4}q_{n+6}^3 + 576H_{n+4}^3 q_{n+6}^2) \\
& + 9z^8(-6f_{n+8}g_{n+12}h_{n+2}^2 + 384f_{8-n}h_{n+2}^3 H_{n+4}q_{n+6} - 384f_8 h_{n+2}H_{n+4}^2 q_{n+6} \\
& \quad - 384f_{8-n}h_{n+2}^2 H_{n+4}^3 + 20f_8 f_{n+8}h_{n+2}^2 H_{n+4} + 120f_8 h_{n+2}^2 q_{n+6}^2 \\
& \quad + 16f_{n+8}^2 h_{n+2}q_{n+6} - f_8^2 h_{n+2}^4 - 8f_{8-n}f_{n+8}h_{n+2}^4 + 144f_{n+8}H_{n+4}q_{n+6}^2 \\
& \quad + 192f_8 H_{n+4}^4 - 4f_{n+8}^2 H_{n+4}^2 + 144g_{n+12}h_{n+2}H_{n+4}q_{n+6} + 144g_{12}h_{n+2}^2 H_{n+4}^2 \\
& \quad - 72g_{12}h_{n+2}^3 q_{n+6} - 96g_{n+12}H_{n+4}^3 + 432q_{n+6}^4) \\
& + 2z^9(-27g_{12}f_{n+8}h_{n+2}^2 - 27f_8 g_{n+12}h_{n+2}^2 + 54f_{n+8}g_{n+12}H_{n+4} \\
& \quad - 1728f_{8-n}h_{n+2}H_{n+4}^2 q_{n+6} + 72f_8^2 h_{n+2}^2 H_{n+4} + 144f_{8-n}f_{n+8}h_{n+2}^2 H_{n+4} \\
& \quad + 864f_{8-n}h_{n+2}^2 q_{n+6}^2 + 144f_8 f_{n+8}h_{n+2}q_{n+6} - 36f_8 f_{8-n}h_{n+2}^4 + 864f_{8-n}H_{n+4}^4 \\
& \quad - 144f_8 f_{n+8}H_{n+4}^2 + 2f_{n+8}^3 + 648g_{12}h_{n+2}H_{n+4}q_{n+6} - 432g_{12}H_{n+4}^3 \\
& \quad + 324g_{n+12}q_{n+6}^2) \\
& + 3z^{10}(-18f_8 g_{12}h_{n+2}^2 + 36g_{12}f_{n+8}H_{n+4} + 96f_8 f_{8-n}h_{n+2}^2 H_{n+4} + 48f_8^2 h_{n+2}q_{n+6} \\
& \quad + 96f_{8-n}f_{n+8}h_{n+2}q_{n+6} - 12f_{8-n}^2 h_{n+2}^4 - 48f_8^2 H_{n+4}^2 - 96f_{8-n}f_{n+8}H_{n+4}^2 \\
& \quad + 4f_8 f_{n+8}^2 + 216g_{12}q_{n+6}^2 + 9g_{n+12}^2) + O(z^{11}). \qquad (5.3)
\end{aligned}
$$

Appendix B

In this appendix we summarize the details of the correspondence between massless matter spectra of $E_8 \times E_8$ heterotic string theory compactified on $K3$ and geometric data of elliptically fibered Calabi-Yau three-fold over Hirzebruch surfaces on which F-theory is compactified.

Let $E_8^{(1)}$ ($E_8^{(2)}$) be the first (second) factor of $E_8 \times E_8$ and $G^{(m)} \times H^{(m)}$ ($m = 1, 2$) be a direct product maximal subgroup of $E_8^{(m)}$ ($m = 1, 2$). We assume that $H^{(1)}$ ($H^{(2)}$) has $12 + n$ ($12 - n$) instantons. We restrict ourselves to the cases where (1) $G^{(m)}$ is simple and simply-laced, and (2) $H^{(m)}$ is semi-simple. The massless spectrum of heterotic string can be computed [29] by the index theorem (3.12).

The TABLE 2 shows the neutral matter spectrum of the heterotic string arising from $E_8^{(1)}$, the corresponding Weierstrass form of the F-theory curve and the independent polynomials which parametrize the curve. The subscripts denote the degrees of the polynomials in z'. For each pair of $G = G^{(1)}$ and $H = H^{(1)}$, the sum of the numbers of the coefficients of the independent polynomials, minus one which takes account of the overall rescaling, always coincides with the number of heterotic singlets obtained by the index theorem, as was verified in (2.6) in section II. A similar result holds for the neutral matter from $E_8^{(2)}$ and the coefficients of the Weierstrass form $\sum_{i=5}^{8} z^i f_{8+(4-i)n}(z')$ and $\sum_{i=7}^{12} z^i g_{12+(6-i)n}(z')$ which determine the singularity at $z = \infty$.

The TABLE 3 shows the spectrum of the charged hypermultiplets. For each (G, H), the leading order term in z of the discriminant of the Weierstrass form factorizes, and the degree in z' of each factor coincides with the number of charged hypermultiplets obtained by the index theorem. What representation occurs is related to the pattern of the singularity enhancement as explained in the text. We have also shown in the last column the corresponding divisor whose intersection number with D_u (the divisor for the $z = 0$ section), determined by the anomaly cancellation conditions (3.21)-(3.25), gives the number of the charged hypermultiplets.

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Table 1. Singularities and string junctions for the unbroken $SU(5)$ case.

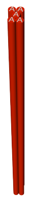
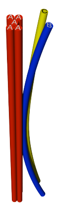
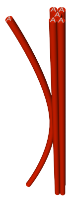
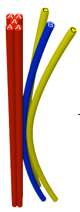
	singularity	7-brane	string junction	$SU(5)$ representation
generic z'	 I_5	$\mathbf{A}^5$	$\pm(a_i - a_j)\ (i<j)$ $+$ Cartan	**24** (gauge symmetry)
locus of h_{n+2}	 I_1^*	$\mathbf{A}^5\mathbf{BC}$	$a_i + a_j - b - c\ (i<j)$	**10**
locus of P_{3n+16}	 I_6	$\mathbf{A}^5\mathbf{A}'$	$a_i - a'\ (i = 1, \ldots, 5)$	**5**
common locus of h_{n+2} and P_{3n+16}	 IV^*	$\mathbf{A}^5\mathbf{BCC}'$	$a_i + a_j - b - c\ (i<j)$ $a_i + a_j - b - c'\ (i<j)$ $\sum_{k=1}^{5} a_k - a_i - 2b - c - c'$ $c - c'$	**10** **10** **5** **1**

Table 2. Heterotic/F-theory duality: Neutral hypermultiplets.

G	H	heterotic/ neutral matter	F-theory curve	indep't polynomials
E_7	$SU(2)$	$(2n+21)\mathbf{1}$	$f_{8+4n}=f_{8+3n}=f_{8+2n}=g_{12+6n}=g_{12+5n}$ $=g_{12+4n}=g_{12+3n}=g_{12+2n}=0$	g_{n+12}, f_{n+8}
E_6	$SU(3)$	$(3n+28)\mathbf{1}$	$f_{8+4n}=f_{8+3n}=f_{8+2n}=g_{12+6n}=g_{12+5n}$ $=g_{12+4n}=g_{12+3n}=0,\quad g_{12+2n}=q_{n+6}^2$	$g_{n+12}, f_{n+8}, q_{n+6}$
$SO(12)$	$SO(4)$	$(2n+18)\mathbf{1}$	$f_{8+4n}=f_{8+3n}=g_{12+6n}=g_{12+5n}=g_{12+4n}=0,$ $f_{8+2n}=-3H_{n+4}^2,\ g_{12+3n}=2H_{n+4}^3,$ $g_{12+2n}=-f_{n+8}H_{n+4},$ $12(g_{12+n}+f_8H_{n+4})H_{n+4}+f_{n+8}^2=q_{n+8}^2$	$q_{n+8}, f_{n+8}, H_{n+4}$ with $n+4$ constraints
$SO(10)$	$SU(4)$	$(4n+33)\mathbf{1}$	$f_{8+4n}=f_{8+3n}=g_{12+6n}=g_{12+5n}=g_{12+4n}=0,$ $f_{8+2n}=-3H_{n+4}^2,\ g_{12+3n}=2H_{n+4}^3,$ $g_{12+2n}=-f_{n+8}H_{n+4}+q_{n+6}^2$	$g_{n+12}, f_{n+8}, q_{n+6}$, H_{n+4}
$SO(8)$	$SO(8)$	$(6n+44)\mathbf{1}$	$f_{8+4n}=f_{8+3n}=g_{12+6n}=g_{12+5n}=g_{12+4n}=0,$ $4f_{8+2n}^3+27g_{12+3n}^2=j_{n+4}^2k_{n+4}^2(j_{n+4}+k_{n+4})^2$	$g_{2n+12}, g_{n+12}, f_{n+8}$, j_{n+4}, k_{n+4}
$SU(6)$	$SU(3)$ $\times SU(2)$	$(3n-r+21)\mathbf{1}$	the same as $SU(5)$ with $h_{n+2}=t_r\tilde{h}_{n+2-r},\ q_{n+6}=u_{r+4}\tilde{h}_{n+2-r},$ $H_{n+4}=t_rq_{n-r+4},$ $f_{n+8}=-12u_{r+4}q_{n-r+4}+t_rp_{n-r+8},$ $g_{n+12}=2u_{r+4}p_{n-r+8}-f_{8-n}h_{n+2}^2+f_8H_{n+4}$	$t_r, \tilde{h}_{n+2-r}, u_{r+4}$, q_{n-r+4}, p_{n-r+8}
$SU(5)$	$SU(5)$	$(5n+36)\mathbf{1}$	$f_{8+4n}=-3h_{n+2}^4, g_{12+6n}=2h_{n+2}^6,$ $g_{12+5n}=-12h_{n+2}^4H_{n+4},$ $f_{8+3n}=12h_{n+2}^2H_{n+4},$ $g_{12+4n}=h_{n+2}^2(12H_{n+4}^2-f_{8+2n}),$ $g_{12+3n}=2f_{8+2n}H_{n+4}+8H_{n+4}^3-f_{8+n}h_{n+2}^2$ $f_{8+2n}=-12H_{n+4}^2+12h_{n+2}q_{n+6}$ $g_{12+2n}=12q_{n+6}^2+2f_{8+n}H_{n+4}-f_8h_{n+2}^2$	h_{n+2}, H_{n+4}, $q_{n+6}, f_{8+n}, g_{12+n}$

$SU(4)$	$SO(10)$	$(8n+51)\mathbf{1}$	$f_{8+4n}=-3h_{n+2}^4, g_{12+6n}=2h_{n+2}^6,$ $g_{12+5n}=-12h_{n+2}^4H_{n+4},$ $f_{8+3n}=12h_{n+2}^2H_{n+4},$ $g_{12+4n}=h_{n+2}^2(12H_{n+4}^2-f_{8+2n}),$ $g_{12+3n}=2f_{8+2n}H_{n+4}+8H_{n+4}^3-f_{8+n}h_{n+2}^2$	$h_{n+2}, H_{n+4}, f_{8+2n},$ $f_{8+n}, g_{12+2n}, g_{12+n}$
$SU(3)$	E_6	$(12n+66)\mathbf{1}$	$f_{8+4n}=-3h_{n+2}^4, g_{12+6n}=2h_{n+2}^6,$ $g_{12+5n}=-12h_{n+2}^3H_{2n+6},$ $g_{12+4n}=12H_{2n+6}^2-h_{n+2}^2f_{8+2n},$ $f_{8+3n}=12h_{n+2}H_{2n+6},$	$h_{n+2}, H_{2n+6}, f_{8+2n},$ $f_{8+n}, g_{12+3n},$ g_{12+2n}, g_{12+n}
$SU(2)$	E_7	$(18n+83)\mathbf{1}$	$f_{8+4n}=-3h_{2n+4}^2, g_{12+6n}=2h_{2n+4}^3,$ $g_{12+5n}=-h_{2n+4}f_{8+3n}$	$h_{2n+4}, f_{8+3n}, f_{8+2n},$ $f_{8+n}, g_{12+4n}, g_{12+3n},$ g_{12+2n}, g_{12+n}

Table 3. Heterotic/F-theory duality: Charged hypermultiplets.

G	H	heterotic/ charged matter	matter locus	singularity enhancement	divisor
E_7	$SU(2)$	$\frac{n+8}{2}\mathbf{56}$	f_{8+n}	$E_7 \to E_8$	$-2K-\frac{3}{2}D_u$
E_6	$SU(3)$	$(n+6)\mathbf{27}$	q_{n+6}	$E_6 \to E_7$	$-3K-2D_u$
$SO(12)$	$SO(4)$	$\frac{n+4}{2}\mathbf{32}$ $(n+8)\mathbf{12}$	H_{n+4} q_{n+8}	$SO(12) \to E_7$ $SO(12) \to SO(14)$	$-K-\frac{1}{2}D_u$ $-4K-3D_u$
$SO(10)$	$SU(4)$	$(n+4)\mathbf{16}$ $(n+6)\mathbf{10}$	H_{n+4} q_{n+6}	$SO(10) \to E_6$ $SO(10) \to SO(12)$	$-2K-D_u$ $-3K-2D_u$
$SO(8)$	$SO(8)$	$(n+4)\mathbf{8}_v$ $(n+4)\mathbf{8}_s$ $(n+4)\mathbf{8}_c$	j_{n+4} k_{n+4} $j_{n+4}+k_{n+4}$	$SO(8) \to SO(10)$ $SO(8) \to SO(10)$ $SO(8) \to SO(10)$	$-2K-D_u$ $-2K-D_u$ $-2K-D_u$
$SU(6)$	$SU(3)$ $\times SU(2)$	$\frac{r}{2}\mathbf{20}$ $(n-r+2)\mathbf{15}$ $(2n+r+16)\mathbf{6}$	t_r $\tilde{h}_{n+2-r}$ $P_{2n+r+16}$	$SU(6) \to E_6$ $SU(6) \to SO(12)$ $SU(6) \to SU(7)$	$-\frac{r}{4}K-\frac{r}{4}D_u$ $(-1+\frac{r}{2})K+\frac{r}{2}D_u$ $-(8+\frac{r}{2})K-(6+\frac{r}{2})D_u$
$SU(5)$	$SU(5)$	$(n+2)\mathbf{10}$ $(3n+16)\mathbf{5}$	h_{n+2} P_{3n+16}	$SU(5) \to SO(10)$ $SU(5) \to SU(6)$	$-K$ $-8K-5D_u$
$SU(4)$	$SO(10)$	$(n+2)\mathbf{6}$ $(4n+16)\mathbf{4}$	h_{n+2} P_{4n+16}	$SU(4) \to SO(8)$ $SU(4) \to SU(5)$	$-K$ $-8K-4D_u$
$SU(3)$	E_6	$(6n+18)\mathbf{3}$	P_{6n+18}	$SU(3) \to SU(4)$	$-9K-3D_u$
$SU(2)$	E_7	$(6n+16)\mathbf{2}$	P_{6n+16}	$SU(2) \to SU(3)$	$-8K-2D_u$